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Three-Valued Brouwer–Zadeh Logic

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A semantic investigation of a particular form of Brouwer–Zadeh logic (three-valued Brouwer–Zadeh logic) is presented and it is shown that this logic can be characterized by means of Kripke-style semantics. Some connections of Brouwer–Zadeh logics with unsharp quantum mechanics are also investigated.

1. INTRODUCTION

Brouwer–Zadeh logics, called also fuzzy-intuitionistic quantum logics, represent nonstandard versions of quantum logic. A characteristic of these logics is a splitting of the connective "not" into two forms of negation: a fuzzy-like negation that gives rise to a paraconsistent behavior, and an intuitionistic-like negation. The fuzzy "not" represents a weak negation that inverts the truth values truth and falsity, satisfies the double negation principle, but generally violates the noncontradiction and the excluded middle principles. The second "not" is a stronger negation, a kind of necessitation of the fuzzy "not." As shown in Cattaneo and Nisticò (1989) and Cattaneo *et al.* (1993), Brouwer–Zadeh logics admit of Hilbert-space exemplifications in the framework of the *unsharp* (or *operational*) approach to quantum mechanics.

In this paper, we will show that a particular kind of Brouwer–Zadeh logic (three-valued Brouwer–Zadeh logic) can be semantically characterized by means of a Kripke semantics. Alternative semantic characterizations can be found in Giuntini (1991, 1992).

2. BROUWER–ZADEH POSET THEORY

A Brouwer–Zadeh poset (*BZ poset*) is a bounded poset with two nonstandard complements linked by an interconnection rule. The first

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complement represents a generalization of the usual complement of fuzzy set theory, whereas the second one is a generalization of the intuitionistic complement. BZ^3 posets are particular BZ posets which satisfy a kind of strong de Morgan law for the intuitionistic complement and a particular equation connecting the two complements.

Definition 2.1. An *involutive bounded poset* (*lattice*) is a structure $\mathscr{P} = \langle P, \leq, ^\perp, \mathbf{1}, \mathbf{0} \rangle$ satisfying the following conditions:

(i) $\langle P, \leq, \mathbf{1}, \mathbf{0} \rangle$ is a partially ordered set (*poset*) (lattice) with *maximum* ($\mathbf{1}$) and *minimum* ($\mathbf{0}$).

(ii) $^\perp$ is a 1-ary operation (the *fuzzy-like complement*) which satisfies the following conditions:

(a) $a^{\perp\perp} = a$.

(b) $\forall a, b \in P$: if $a \leq b$, then $b^\perp \leq a^\perp$.

Two elements a, b of an involutive bounded poset are said to be *orthogonal* ($a \perp b$) iff $a \leq b^\perp$. The *inf* and the *sup* of two elements a, b, when they exist, will be denoted by $a \sqcap b$ and $a \sqcup b$, respectively.

Definition 2.2. A *regular involutive bounded poset* (*lattice*) is an involutive bounded poset (lattice) $\mathscr{P} = \langle P, \leq, ^\perp, \mathbf{1}, \mathbf{0} \rangle$ satisfying the following condition:

$\forall a, b \in P$: if $a \perp a$ and $b \perp b$, then $a \perp b$ (*regularity condition*)

It is easy to see that an involutive bounded lattice $\mathscr{P}$ is regular iff it satisfies the *Kleene condition*, i.e., $\forall a, b \in P$: $a \sqcap a^\perp \leq b \sqcup b^\perp$.

Definition 2.3. An *orthoposet* (*ortholattice*) is an involutive bounded poset (lattice) $\mathscr{P}$ which satisfies the following condition: $\forall a \in P$: $a \sqcap a^\perp = \mathbf{0}$.

Definition 2.4. A *Brouwer–Zadeh poset* (*BZ poset*) (*BZ lattice*) is a structure $\mathscr{P} = \langle P, \leq, ^\perp, ^\sim, \mathbf{1}, \mathbf{0} \rangle$ which satisfies the following conditions:

(i) $\mathscr{P} = \langle \mathscr{P}, \leq, ^\perp, \mathbf{1}, \mathbf{0} \rangle$ is a regular involutive bounded poset (lattice).

(ii) $\sim$ is a 1-ary operation (the *intuitionistic-like complement*) which satisfies the following conditions $\forall a, b \in P$:

(a) $a \leq a^{\sim\sim}$.

(b) If $a \leq b$, then $b^\sim \leq a^\sim$.

(c) $a \sqcap a^\sim = \mathbf{0}$.

(iii) $a^{\sim\perp} = a^{\sim\sim}$.

Lemma 2.1. Let $\mathscr{P}$ be a BZ poset. The following conditions hold:

(i) $\forall a \in P$: $a^\sim \leq a^\perp$.

(ii) $\forall a, b \in P$: if $a \sqcup b$ exists in P, then $a^{\sim} \sqcap b^{\sim}$ exists in P and $(a \sqcup b)^{\sim} = a^{\sim} \sqcap b^{\sim}$.

Example 2.1. Let $L := [0, 1] \subset \mathbb{R}$ and let $\leq$ be the natural order of $\mathbb{R}$ restricted to $[0, 1]$. We can define on $[0, 1]$ the following operations for all $a \in L$:

(i) $a^{\perp} = 1 - a$.

(ii)

$$a^{\sim} = \begin{cases} 1 & \text{if } \quad a = 0 \\ 0 & \text{otherwise} \end{cases}$$

The structure $\mathfrak{L}_{[0,1]} := \langle L_{[0,1]}, \leq, {}^{\perp}, {}^{\sim}, 1, 0 \rangle$ turns out to be a BZ lattice, where $a \sqcap b := \min(\{a, b\})$ and $a \sqcup b := \max(\{a, b\})$.

Example 2.2. Let $E(\mathfrak{H})$ be the class of all *effects* of a Hilbert space $\mathfrak{H}$. An effect is a linear bounded operator E s.t. $\mathbb{0} \leq E \leq \mathbb{1}$, where $\mathbb{0}$ and $\mathbb{1}$ are the null and the identity operators, respectively. Let us define the following relation and operations for all $E, F \in E(\mathfrak{H})$:

(i) $E \leq F$ iff $\forall \varphi \in \mathfrak{H}$: $(E\varphi, \varphi) \leq (F\varphi, \varphi)$.

(ii) $E^{\perp} = \mathbb{1} - E$.

(iii) $E^{\sim} = P_{\mathrm{Kern}(E)}$, where $P_{\mathrm{Kern}(E)}$ is the projector associated with the closed subspace $\mathrm{Kern}(E) := \{\varphi \in \mathfrak{H} | E\varphi = 0\}$.

The structure $\mathfrak{E}(\mathfrak{H}) = \langle E(\mathfrak{H}, \leq, {}^{\perp}, {}^{\sim}, \mathbb{0}, \mathbb{1} \rangle$ turns out to be a BZ poset which is not a lattice.

Lemma 2.2. Let $\mathscr{P}$ be a BZ poset. Then, the following conditions are equivalent $\forall a \in P$:

(i) $a = a^{\sim\sim}$.

(ii) $a^{\sim} = a^{\perp}$.

(iii) $a = a^{\sim\perp}$.

(iv) $a = a^{\perp\sim}$.

Lemma 2.3. Let $\mathscr{P} = \langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0} \rangle$ be a BZ poset. Then the set $P_e := \{a \in P / a = a^{\sim\sim}\}$ is not empty, since $\mathbf{0}, \mathbf{1} \in P_e$ and, moreover:

(i) $a^{\sim} = a^{\perp}, \forall a \in P_e$. The set P_e endowed with the restriction of the partial order $\leq$ defined on P is an orthoposet with respect to the orthocomplementation $^{\sim}: P_e \to P_e$.

(ii) If $\mathscr{P}$ is a lattice, then P_e is closed under the *inf* and the *sup* operations of P.

The elements of P_e are called *exact elements* of $\mathscr{P}$ and the elements of P/P_e are called *fuzzy elements* of $\mathscr{P}$.

In Giuntini (1992) it is proved that the MacNeille completion of every BZ poset is a complete BZ lattice. Therefore, every BZ poset can be embedded into a complete lattice.

Definition 2.5. Let $\mathscr{P} = \langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ be a BZ poset. An element $a \in P$ is said to be a *half element* of $\mathscr{P}$ iff $a^{\perp} = a$.

It is easy to see that if a BZ poset has a half element, then this is unique. Such an element will be denoted by 1/2.

The set of all effects of a Hilbert space is a BZ poset with the half element.

Definition 2.6. A *BZ* poset* (lattice) is a BZ poset (lattice) $\mathscr{P} = \langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ which satisfies the following conditions $\forall a, b \in P$:

(*) If $a^{\perp\sim} \leq b$ and $a \leq b^{\sim\sim}$, then $a \leq b$.

If $\mathscr{P}$ is a BZ lattice, then condition (*) is equivalent to the following (equational) condition: $\forall a, b \in P$: $a \sqcap b^{\sim\sim} \leq a^{\perp\sim} \sqcup b$.

If $\mathscr{P}$ is a BZ* poset, then we have $\forall a \in P$: $a^{\perp\sim} = \mathbf{0}$ iff $a \leq a^{\perp}$.

One can easily show that both the BZ lattice of Example 2.1 and the BZ poset of Example 2.2 *do not* satisfy condition (*).

Definition 2.7. A *de Morgan BZ poset* is a BZ poset $\mathscr{P} = \langle P, \leq, {}^{\perp}, {}^{\sim}, \mathbf{1}, \mathbf{0}\rangle$ which satisfies the following condition $\forall a, b \in P$:

(DM) If $a \sqcap b$ exists in P, then $a^{\sim} \sqcup b^{\sim}$ exists in P and $(a \sqcap b)^{\sim} = a^{\sim} \sqcup b^{\sim}$.

As one can readily see, the BZ lattice of Example 2.1 satisfies condition (DM).

Theorem 2.1. Let $\mathfrak{H}$ be a finite-dimensional Hilbert space. The set of all effects of $\mathfrak{H}$ is a de Morgan BZ poset.

Proof. See Cattaneo and Giuntini (1993).

It is an open question whether the BZ poset of all effects of an infinite dimensional Hilbert space satisfies condition (DM). Moreover, it is not known whether every de Morgan BZ lattice can be embedded into a complete de Morgan BZ lattice. Further, it is not known whether the de Morgan BZ poset of all effects of a finite-dimensional Hilbert space can be embedded into a complete de Morgan BZ poset.

Definition 2.8. A *BZ^3 poset* (*BZ^3 lattice*) is a BZ poset (lattice) $\mathscr{P}$ which satisfies conditions (*) and (DM).

As proved in Giuntini (1992), conditions (*) and (DM) are independent.

3. ORTHO-PAIR SEMANTICS

The language of *Brouwer–Zadeh* logics contains a denumerable set of *sentential letters* $p_1, p_2, \ldots, p_n, \ldots$ and the (primitive) connectives $\wedge$ (and), $\neg$ (fuzzy negation), and $\sim$ (intuitionistic negation). We will use $p, q, r, \ldots$, as metavariables ranging over sentential letters and $\alpha, \beta, \gamma, \ldots$ as metavariables for formulas. The disjunctive connective $\vee$ is defined by means of a metalinguistic definition:

$$\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$$

In the following, we will consider a particular form of Brouwer–Zadeh logic, the so-called three-valued Brouwer–Zadeh logic (BZL^3). Such a logic turns out to be properly stronger than weak Brouwer–Zadeh logic (BZL) (the logic which is algebraically characterized by the equational class of all BZ lattices (Cattaneo *et al.*, 1993)). In this section, we will present a semantic characterization of BZL^3 based on a kind of many-valued possible world semantics (called *ortho-pair semantics*) first proposed by Cattaneo and Nisticò (1989). The intuitive idea can be sketched as follows: one supposes that interpreting a language means associating to any sentence two *domains of certainty*: the domain of possible worlds where the sentence certainly holds and the domain of possible worlds where the sentence certainly does not hold. All the other worlds are supposed to associate an intermediate truth-value (*indetermined*) to our sentence. Differently from the standard Kripkean behavior, the positive domain of a given sentence does not generally determine the negative domain of the same sentence. As a consequence, propositions are here identified with particular pairs of sets of worlds, rather than with particular sets of worlds (as happens in the usual possible worlds semantics).

The models of this semantics will be called models with positive and negative domains (shortly, *ortho-pair models*). A Hilbert-space exemplification of this semantics can be found in Cattaneo *et al.* (1993).

Definition 3.1. A *preclusivity space* is a pair $\mathcal{M} = \langle I, \# \rangle$, where I is a nonempty set and $\#$ is an irreflexive and symmetric binary relation on I. A *simple proposition* of a preclusivity space $\mathcal{M} = \langle I, \# \rangle$ is a subset A of I such that $A = A^{\#\#}$, where

$$A^{\#} := \{i \in I | \forall j \in A : i \# j\}$$

Let $P(I)$ be the set all simple propositions. Then, the structure

$$\mathcal{P}(I) := \langle P(I), \subseteq, {}^{\#}, \varnothing, I \rangle$$

is a complete ortholattice. Let us indicate by $^{\#}, (\sqcap), (\sqcup)$ the lattice operations defined on $P(I)$. $(\bigvee), (\bigwedge)$ will represent the infinitary lattice operations.

Definition 3.2. A *possible proposition* of $\mathcal{M}$ is any pair $\langle A_1, A_0 \rangle$, where A_1, A_0 are simple propositions such that $A_1 \perp A_0$ (in other words, A_1 and A_0 are preclusive).

A possible proposition $\langle A_1, A_0 \rangle$ is called *exact* iff $A_0 = A_1^{\#}$ (in other words, A_0 is maximal).

The following operations and relations are defined on the set $T(I)$ of all possible propositions:

(i) The fuzzy complement:

$$\langle A_1, A_0 \rangle^{\perp} = \langle A_0, A_1 \rangle$$

(ii) The intuitionistic complement:

$$\langle A_1, A_0 \rangle^{\sim} = \langle A_0, A_0^{\#} \rangle$$

(iii) The propositional conjunction:

$$\langle A_1, A_0 \rangle \sqcap \langle B_1, B_0 \rangle = \langle A_1(\sqcap)B_1, A_0(\sqcup)B_0 \rangle$$

(iv) The propositional disjunction:

$$\langle A_1, A_0 \rangle \sqcup \langle B_1, B_0 \rangle = \langle A_1(\sqcup)B_1, A_0(\sqcap)B_0 \rangle$$

(v) The infinitary conjunction:

$$\bigwedge_n \{\langle A_1^n, A_0^n \rangle\} = \left\langle \left(\bigwedge\right)_n \{A_1^n\}, \left(\bigvee\right)_n \{A_0^n\} \right\rangle$$

(vi) The infinitary disjunction:

$$\bigvee_n \{\langle A_1^n, A_0^n \rangle\} = \left\langle \left(\bigvee\right)_n \{A_1^n\}, \left(\bigwedge\right)_n \{A_0^n\} \right\rangle$$

(vii) The order-relation:

$$\langle A_1, A_0 \rangle \sqsubseteq \langle B_1, B_0 \rangle \quad \text{iff} \quad A_1 \subseteq B_1 \quad \text{and} \quad B_0 \subseteq A_0$$

(viii) The absurd proposition:

$$\mathbf{0} = \langle \varnothing, I \rangle$$

(ix) The trivial proposition:

$$\mathbf{1} = \langle I, \varnothing \rangle.$$

Then, one can prove the following theorems:

Theorem 3.1. The structure $\mathcal{T}(I) = \langle T(I), (\wedge), (\vee), ^{\perp}, ^{\sim}, \langle \varnothing, I\rangle, \langle I, \varnothing\rangle\rangle$ is a complete BZ^3 lattice with the half element $\langle \varnothing, \varnothing\rangle$.

Proof. See Cattaneo and Nisticò (1989).

Theorem 3.2. Every BZ^3 poset can be embedded into the complete BZ^3 lattice of all possible propositions of a preclusivity space.

Proof. See Giuntini (1992).

By Theorems 3.1 and 3.2 one can conclude that every BZ^3 poset can be embedded into a complete BZ^3 lattice.

Definition 3.3. An *ortho-pair model* is a system $\mathcal{M} = \langle I, \#, \Pi, v\rangle$, where:

(i) $\langle I, \#\rangle$ is a preclusivity space.

(ii) Π is a subset of all possible propositions of $\langle I, \#\rangle$ which is closed under $^{\perp}$, $^{\sim}$, $\sqcup$, $\sqcap$, and $\mathbf{0} := \langle \varnothing, I\rangle$.

(iii) v is an *interpretation-function* that maps formulas into Π according to the following conditions:

(a) $v(p) \in \Pi$, for all sentential letters p.

(b) $v(\beta \wedge \gamma) = v(\beta) \sqcap v(\gamma)$.

(c) $v(\neg \beta) = v(\beta)^{\perp}$.

(d) $v(\sim \beta) = v(\beta)^{\sim}$.

Definition 3.4. A formula α is *valid* in an ortho-pair model $\mathcal{M} = \langle I, \#, \Pi, v\rangle$ ($\vDash_{\mathcal{M}} \alpha$) iff $v(\alpha) = \mathbf{1} = \langle I, \varnothing\rangle$.

A formula α is a *logical truth* of BZL^3 ($\vDash_{BZL^3} \alpha$) iff for any ortho-pair model $\mathcal{M}$, $\vDash_{\mathcal{M}} \alpha$.

Definition 3.5. Let T be a set of formulas and let $\mathcal{M} = \langle I, \#, \Pi, v\rangle$ be an ortho-pair model. We say that α is an *$\mathcal{M}$-consequence* of T ($T \vDash_{\mathcal{M}} \alpha$) iff $\forall \langle X_1, X_0\rangle \in \Pi$: if $\forall \beta \in T$: $\langle X_1, X_0\rangle \sqsubseteq v(\beta)$, then $\langle X_1, X_0\rangle \sqsubseteq v(\alpha)$.

α is a *logical consequence* of T *in the ortho-pair semantics* of BZL^3 ($T \vDash^{o}_{BZL^3} \alpha$) iff for any ortho-pair model $\mathcal{M}$: $T \vDash_{\mathcal{M}} \alpha$.

BZL^3 can be axiomatized and a strong completeness theorem (based on the ortho-pair semantics) can be proved (Cattaneo *et al.*, 1993).

One can construct also an algebraic semantics for BZL^3, based on the equational class of all BZ^3 lattices (Giuntini, 1992). By Theorem 3.2, one can prove that the ortho-pair semantics and the algebraic semantics strongly characterize the same logic.

In Giuntini (1993) we have developed a filtration technique for ortho-pair semantics and we have proved that BZL^3 has the finite model property; consequently, BZL^3 is decidable. The finite model property for

BZL instead is proved by means of filtration techniques based on the Kripkean semantics (Giuntini, 1992).

4. KRIPKEAN SEMANTICS FOR BZL^3

In this section, we will present a Kripkean semantics for BZL^3 and will prove a representation theorem for BZ^3 lattices based on the notion of BZ^3 frame. Finally, we will show that the Kripkean semantics for BZL^3 and the ortho-pair semantics strongly characterize the same logic.

Differently from the usual Kripke frames, a BZ^3 frame contains two accessibility relations($\not\perp$ and $\not\dot\perp$) and two unary operations ($\Box$ and *). The accessibility relation $\not\perp$ (which determines the fuzzy-like complement) is stronger that the accessibility relation $\not\dot\perp$ (which determines the intuitionistic-like complement). Moreover, the two accessibility relations are connected by means of the two unary operations. For any world i there exists a "twin-world" j which represents the "possibility-world" associated with i. "Twin" means here that the two worlds i and j cannot be distinguished by means of the weak accesibility relation $\not\dot\perp$. The unary operation $\Box$ can be interpreted as that function which "extracts" from any world i, its "necessity-region." The operation * can be intuitively interpreted as a function which extracts the "$\perp$-contradictory region" of the world i itself.

Definition 4.1. A *BZ^3 frame* is a system $\mathscr{F} = \langle W, \perp, \dot\perp, \Box, *, \lambda\rangle$, where W is a nonempty set (the set of *possible worlds*) and λ (the *absurd world*) is an element not belonging to W; moreover, $\perp$ and $\dot\perp$ are two binary relations on $W^\lambda := W \cup \{\lambda\}$, whereas $\Box$ and * are two unary operations on W^λ. The following conditions are required to hold:

(i) $\perp$ is symmetric.

(ii) $\forall i \in W, \exists j \in W$ s.t. $i \not\perp j$, where $i \not\perp j$ means not $i \perp j$.

(iii) $\forall i \in W$: $i \not\dot\perp i$, where $i \not\dot\perp i$ means not $i \dot\perp i$.

(iv) $\forall i \in W^\lambda$: $i \dot\perp \lambda$.

(v) $\dot\perp$ is symmetric.

(vi) $\forall i, j \in W^\lambda$: if $i \dot\perp j$, then $i \perp j$.

(vii) $\forall i \in W^\lambda, \exists j \in W^\lambda$ s.t. $\forall k \in W^\lambda$ the following conditions are satisfied:

(a) $i \dot\perp k$ iff $j \dot\perp k$.

(b) If $j \perp k$, then $i \dot\perp k$.

(viii) $\forall i \in W^\lambda$: $i \perp i$ iff $\Box\, i = \lambda$.

(ix) $\forall i \in W^\lambda$: if $i \perp j$, then $\Box\, i \dot\perp j$.

(x) $\forall i, j \in W^\lambda$: if $\Box\, i \dot\perp j$ and $i \dot\perp \Box j$, then $i \perp j$.

(xi) $\forall i, j \in W^\lambda$: if $i^* \dot\perp \Box j$, then $i^* \perp j$.

(xii) $\forall i, j \in W^\lambda$: $i^* \dot\perp j$ iff $i \dot\perp j$.

It should be noted that the regularity condition for $\perp$ ($i \perp i$ and $j \perp j$ implies $i \perp j$) can be derived. Let us suppose that $i \perp i$ and $j \perp j$. By condition (viii), $\Box i = \Box j = \lambda$. By condition (iv), $\Box i \mathrel{\underline{\perp}} j$ and $i \mathrel{\underline{\perp}} \Box j$. Hence, by (x), $i \perp j$.

Let $\mathscr{F} = \{W, \perp, \mathrel{\underline{\perp}}, \Box, {}^*, \lambda\rangle$ be a BZ^3 frame and let $A \subseteq W^\lambda$. We can define the following two operations:

$$A^\perp = \{i \in W^\lambda | \forall j \in A : i \perp j\}, \qquad A^\sim = \{i \in W^\lambda \mid \forall j \in A\}$$

Let

$$P(\mathscr{F}) := \{A \subseteq W^\lambda | A = A^{\perp\perp}\}$$

(the set of all $\perp$-*regular subsets* of $\mathscr{F}$). In Giuntini (1991) we have proved that the structure $\mathscr{P}(\mathscr{F}) = \langle P(\mathscr{F}), \subseteq, {}^\perp, {}^\sim, \{\lambda\}, W^\lambda\rangle$ is a complete BZ lattice, where $\{\lambda\}$ and W^λ are the minimum and the maximum of the lattice, respectively. Given two $\perp$-regular subsets A, B, the inf of A, B is $A \cap B$ and the sup (denoted by $A \sqcup B$) is $(A \cup B)^{\perp\perp}$.

Lemma 4.1. Let $\mathscr{F} = \langle W, \perp, \mathrel{\underline{\perp}}, \Box, {}^*, \lambda\rangle$ be a BZ^3 frame. Then, the following properties hold true $\forall A, B \in P(\mathscr{F})$ and $\forall i \in W^\lambda$:

(a) If $i \in A^\perp$, then $\Box i \in A^\sim$.

(b) If $i \in A^{\sim\sim}$, then $i^* \in A$.

Proof. (a) Let us suppose that $i \in A^\perp$ and $\Box i \notin A^\sim$. Then, $\exists j \in A$ s.t. $\Box i \not\mathrel{\underline{\perp}} j$. By hypothesis, $i \perp j$ and therefore, by condition (ix), $\Box i \mathrel{\underline{\perp}} j$, contradiction.

(b) Let us suppose that $i \in A^{\sim\sim}$; then, by condition (xii), $i^* \in A^{\sim\sim}$. Let us suppose, by contradiction, that $i^* \notin A$. Then, $\exists j \in A^\perp$ s.t. $i^* \not\perp j$. By the result previously proved, $\Box j \in A^\sim$, so that $\Box j \mathrel{\underline{\perp}} i^*$; hence, by condition (xi), $i^* \perp j$, contradiction. ■

Lemma 4.2. $\forall A, B \in P(\mathscr{F})$: $(A \cap B)^\sim = A^\sim \sqcup B^\sim$.

Proof. The inclusion $A^\sim \sqcup B^\sim \subseteq (A \cap B)^\sim$ is trivial.

Let us suppose that $i \in (A \cap B)^\sim$. Let us suppose, by contradiction, that $i \notin A^\sim \sqcup B^\sim$. Then $\exists j \in (A^\sim \sqcup B^\sim)^\perp = A^{\sim\perp} \cap B^{\sim\perp} = A^{\sim\sim} \cap B^{\sim\sim}$ s.t. $i \not\perp j$. By Lemma 4.1(b), $j^* \in A \cap B$. By hypothesis, $i \mathrel{\underline{\perp}} j^*$, so that, by conditions (xii) and (vi), we get $i \perp j$, contradiction. ■

Theorem 4.1. $\mathscr{P}(\mathscr{F})$ is a complete BZ^3 lattice.

Proof. As previously mentioned, $\mathscr{P}(\mathscr{F})$ is a complete BZ lattice and by Lemma 4.2, it satisfies the strong de Morgan law. Thus, to prove that $\mathscr{P}(\mathscr{F})$ is a BZ^3 lattice it suffices to show that $\mathscr{P}(\mathscr{F})$ satisfies condition (*) of Definition 2.6. Let us suppose that $A^{\perp\sim} \subseteq B$ and $A \subseteq B^{\sim\sim}$; we want to show that $A \subseteq B$. Let us suppose that $i \in A$ but $i \notin B$. Then $\exists j \in B^\perp$ s.t. $i \not\perp j$. By hypothesis, $A^{\perp\sim} \subseteq B$, so that $B^\perp \subseteq A^{\perp\sim\perp} = A^{\perp\sim\sim}$. Then, $j \in A^{\perp\sim\sim}$, since $j \in B^{\perp}$. Now, $i \in A$ and therefore, by Lemma 4.1(a), $\Box i \in A^{\perp\sim}$. Hence, $\Box i \not\perp j$. From $j \in B^{\perp}$, it follows, by Lemma 4.1(a), that $\Box j \in B^{\sim}$. But $i \in B^{\sim\sim}$ because $A \subseteq B^{\sim\sim}$; hence, $i \not\perp \Box j$. Thus, $\Box i \not\perp j$ and $i \not\perp \Box j$. By condition (x), we get $i \perp j$, contradiction. ∎

Lemma 4.3. Let $\mathscr{F} = \langle W, \perp, \not\perp, \Box, {}^*, \lambda \rangle$ be a BZ^3 frame. If W^{λ} contains an element u (the indeterminate world) s.t. $u \perp u$ and $\forall i \in W^{\lambda}$: $i \perp i$ iff $i \perp u$, then $\mathscr{P}(\mathscr{F})$ has the half element $1/2 := \{u\}^{\perp}$.

Proof. First of all note that $\{u\}^{\perp}$ is an element of $\mathscr{P}(\mathscr{F})$. We now show that $\{u\}^{\perp} = \{u\}^{\perp\perp}$. Let us suppose that $i \in \{u\}^{\perp\perp}$. Then, $\forall j \in \{u\}^{\perp}$: $i \perp j$. By hypothesis, $u \perp u$, so that $i \perp u$, i.e., $i \in \{u\}^{\perp}$.

Let us suppose that $i \in \{u\}^{\perp}$; then, $i \perp u$. Let us suppose that $j \in \{u\}^{\perp}$. We want to show that $i \perp j$. Now, $i \perp u$ and $j \perp u$, so that, by definition of the element u, $i \perp i$ and $j \perp j$. By the regularity condition for $\perp$, we get $i \perp j$. ∎

Theorem 4.1 shows that every BZ^3 frame determines a complete BZ^3 lattice. The next theorem shows that every BZ^3 lattice is determined by some BZ^3 frame; consequently, every BZ^3 lattice is isomorphic to a BZ^3 lattice of sets.

Theorem 4.2. Every BZ^3 lattice is embeddable into the BZ^3 lattice of all $\perp$-regular subsets of some BZ^3 frame.

Proof. Let $\mathscr{L}$ be a BZ^3 lattice. We can assume that $\mathscr{L}$ has the half element 1/2; indeed, even if L does not contain 1/2, $\mathscr{L}$ can be embedded, via Theorem 3.2, into a BZ^3 lattice with the half element.

Let us consider the system $\mathscr{F} = \langle W, \perp, \not\perp, \Box, {}^*, \lambda \rangle$ defined as follows:

(a) W is the set of all proper filters of $\mathscr{L}$.
(b) λ is the trivial filter L.
(c) $\forall F, G \in W^{\lambda}$: $F \perp G$ iff $\exists a \in L$: $a \in F$ and $a^{\perp} \in G$.
(d) $\forall F, G \in W^{\lambda}$: $F \not\perp G$ iff $\exists a \in L$: $a \in F$ and $a^{\sim} \in G$.
(e) $\forall F \in W^{\lambda}$: $\Box F = \{a \in L \mid \exists b \in F : b^{\perp\sim} \leq a\}$.
(f) $\forall F \in W^{\lambda}$: $F^* = \{a \in L \mid \exists b \in F : b^* := b \sqcap \frac{1}{2} \leq a\}$.

We want to show that $\mathscr{F}$ is a BZ^3 frame. An easy computation shows that the operations $\Box$ and $*$ are well defined, i.e., $\forall F \in W$: $\Box F$, $F^* \in W$. Moreover, $\forall F \in W^{\lambda}$: $F \subseteq \Box F$ and $F \subseteq F^*$.

Conditions (i)–(vi) are trivially verified. For the proof of condition (vii), see Giuntini (1991).

(viii) Let us suppose that $F \perp F$. Then, $\exists a \in L$ s.t. $a \in F$ and $a^{\perp} \in F$. Clearly, $a^{\perp\sim} \in \Box F$ and $a^{\perp} \in \Box F$, since $F \subseteq \Box F$: Thus, $\mathbf{0} = a^{\perp\sim} \sqcap a^{\perp} \in \Box F$; hence, $\Box F = L = \lambda$. Let us suppose that $\Box F = \lambda$, i.e., $\mathbf{0} \in \Box F$. Then, $\exists b \in F$

s.t. $b^{\perp\sim} = \mathbf{0}$. By condition (*) of Definition 2.6, it follows that $b \le b^{\perp}$, so that $b^{\perp} \in F$, i.e., $F \perp F$.

(ix) Let us suppose $F \perp G$. Then, $\exists a \in L$ s.t. $a \in F$ and $a^{\perp} \in G$. Since $F \subseteq \Box F$, we have that $a \in \Box F$ and therefore, $a^{\perp\sim} \in \Box F$; hence, $\Box F \not\perp G$.

(x) Let us suppose that $\Box F \not\perp G$ and $F \not\perp \Box G$. Then, $\exists a, b \in L$ s.t. $a \in \Box F$, $a^{\sim} \in G$, $b \in F$, and $b^{\sim} \in \Box G$. By definition of $\Box F$, $\exists c \in F$ s.t. $c^{\perp\sim} \le a$ and $\exists d \in G$ s.t. $d^{\perp\sim} \le b^{\sim}$. Thus, $a^{\sim} \le c^{\perp\sim\sim}$ so that $c^{\perp\sim\sim} \in G$. Now, $b \le b^{\sim\sim} \le d^{\perp\sim\sim}$, so that $d^{\perp\sim\sim} \in F$. Hence, $c \sqcap d^{\perp\sim\sim} \in F$ and $c^{\perp\sim\sim} \sqcap d \in G$. By condition (*), $c \sqcap d^{\perp\sim\sim} \le c^{\perp\sim} \sqcup d^{\perp}$. Thus, $(c^{\perp\sim\sim} \sqcap d)^{\perp} = c^{\perp\sim} \sqcup d^{\perp} \in F$. This means that $F \perp G$.

(xi) Let us suppose that $F^* \not\perp \Box G$. Then, $\exists a \in L$ s.t. $a \in F^*$ and $a^{\sim} \in \Box G$. Then, $\exists b, c \in L$ s.t. $b \in F, b^* \le a, c \in G$, and $c^{\perp\sim} \le a^{\sim}$. Now, $a^{\sim} \le (b^*)^{\sim}$, so that $b^* \le c^{\perp\sim\sim}$. Since $(b^*)^{\perp\sim} = (b \sqcap \frac{1}{2})^{\perp\sim} = \mathbf{0} \le c^{\perp}$, we have, by condition (*), that $b^* \le c^{\perp}$. Thus, $(b^*)^{\perp} \in G$. But $b^* \in F^*$, since $F \subseteq F^*$; hence, $F \perp G$.

(xii) Let us suppose that $F^* \not\perp G$. Then, $\exists a \in L$ s.t. $a \in F^*$ and $a^{\sim} \in G$. Then, $\exists b \in L$ s.t. $b \in F$ and $b^* \le a$. Thus, $a^{\sim} \le (b^*)^{\sim} := (b \sqcap \frac{1}{2})^{\sim} = b^{\sim} \sqcup \frac{1}{2}^{\sim} = b^{\sim} \sqcup \mathbf{0} = b^{\sim}$. Hence, $b^{\sim} \in G$, so that $F \not\perp G$.

Let us suppose that $F \not\perp G$. Then, $F^* \not\perp G$, since $F \subseteq F^*$.

We can now define the map $h: L \mapsto 2^{W^{\lambda}}$ in the following way: $\forall a \in L$: $h(a) = \{F \in W^{\lambda} \mid a \in F\}$. It is not hard to see that $h(a^{\perp}) = h(a)^{\perp}$, i.e., h maps L into $\mathscr{P}(\mathscr{F})$, the set of all $\perp$-regular subsets of the BZ^3 frame $\mathscr{F}$. By standard techniques, one can show that h is an embedding of $\mathscr{L}$ into $P(\mathscr{F})$. ■

Definition 4.2. A *Kripke model* for BZL^3 is a system $\mathscr{K} = \langle W, \perp, \not\perp, \Box, *, \lambda, \Omega, \rho \rangle$, where

(i) $\mathscr{F} := \langle W, \perp, \not\perp, \Box, *, \lambda \rangle$ is BZ^3 frame.

(ii) Ω is a subset of $P(\mathscr{F})$ which is closed under $\cap$, $^{\perp}$, $^{\sim}$ and $\mathbf{0} := \{\lambda\}$.

(iii) ρ is a valuation function which maps formulas into Ω according to the following conditions:

(a) $\rho(p) \in \Omega$, for all sentential letters p.

(b) $\rho(\beta \wedge \gamma) = \rho(\beta) \cap \rho(\gamma)$.

(c) $\rho(\neg\beta) = \rho(\beta)^{\perp}$.

(d) $\rho(\sim\beta) = \rho(\beta)^{\sim}$.

The other semantic definitions are given as in ortho-pair semantics. We will write $T \models^{K}_{\mathrm{BZL}^3} \alpha$ to denote that α is a logical consequence of T in the Kripke semantics for BZL^3.

We can now prove that the Kripke semantics and the ortho-pair semantics strongly characterize the same logic.

Theorem 4.3.

$$T \models^K \alpha \quad \text{iff} \quad T \models^O \alpha$$

The proof of this theorem is a direct consequence of the following two lemmas:

Lemma 4.4. Every Kripke model $\mathscr{K} = \langle W, \perp, \perp\!\!\!\!\sim, \Box, {}^*, \lambda, \Omega, \rho \rangle$ can be transformed into an ortho-pair model $\mathscr{M}^{\mathscr{K}} = \langle I, \#, \Pi, v \rangle$ s.t.

$$T \models_{\mathscr{K}} \alpha \quad \text{iff} \quad T \models_{\mathscr{M}^{\mathscr{K}}} \alpha$$

Lemma 4.5. Every ortho-pair model $\mathscr{M} = \langle I, \#, \Pi, v \rangle$ can be transformed into a Kripke model $\mathscr{K}^{\mathscr{M}} = \langle W, \perp, \perp\!\!\!\!\sim, \Box, {}^*, \lambda, \Omega, \rho \rangle$ s.t.

$$T \models_{\mathscr{M}} \alpha \quad \text{iff} \quad T \models_{\mathscr{K}^{\mathscr{M}}} \alpha$$

Sketch of the Proof of Lemma 4.4. By Theorem 4.1, Ω is a BZ^3 sublattice of the complete BZ^3 lattice $\mathscr{P}(\mathscr{F})$ of all $\perp$-regular subsets of the BZ^3 frame $\mathscr{F} := \langle W, \perp, \perp\!\!\!\!\sim, \Box, {}^*, \lambda \rangle$. Thus, the Kripke model $\mathscr{K}$ can be transformed into a Kripke model $\mathscr{K}^c$ by replacing Ω with $P(\mathscr{F})$. Clearly, $T \models_{\mathscr{K}} \alpha$ iff $T \models_{\mathscr{K}^c} \alpha$. By Theorem 3.2, $\mathscr{P}(\mathscr{F})$ can be embedded into the complete BZ^3 lattice $\mathscr{T}(I)$ of all possible propositions (ortho-pairs) of a preclusivity space $\langle I, \# \rangle$. Let $k: P(\mathscr{F}) \to T(I)$ be such an embedding. Take Π as the range of k. Define $v(\gamma) = k(v(\gamma))$, for any formula γ. One can easily show that the system $\mathscr{M}^{\mathscr{K}} = \langle I, \#, \Pi, v \rangle$ is a "good" ortho-pair model s.t. $T \models_{\mathscr{K}^c} \alpha$ iff $T \models_{\mathscr{M}^{\mathscr{K}}} \alpha$. ■

Sketch of the Proof of Lemma 4.5. By Theorem 3.1, Π is a BZ^3 sublattice of the complete BZ^3 lattice $\mathscr{T}(I)$ of all possible propositions of $\langle I, \# \rangle$. Let us consider the ortho-pair model $\mathscr{M}^c$ which is obtained from $\mathscr{M}$ by replacing Π with $T(I)$. Clearly, $T \models_{\mathscr{M}} \alpha$ iff $T \models_{\mathscr{M}^c} \alpha$.

Let us define the system $\mathscr{K}^{\mathscr{M}} = \langle W, \perp, \perp\!\!\!\!\sim, \Box, {}^*, \lambda, \Omega, \rho \rangle$ in the following way:

(i) W is the $T(I) \backslash \{\langle \varnothing, W \rangle\}$.

(ii) $\forall \langle A_1, A_0 \rangle, \langle B_1, B_0 \rangle \in W$:

(a) $\langle A_1, A_0) \perp \langle B_1, B_0 \rangle$ iff $\langle A_1, A_0 \rangle \sqsubseteq \langle B_1, B_0 \rangle^{\perp}$.

(b) $\langle A_1, A_0 \rangle \perp\!\!\!\!\sim \langle B_1, B_0 \rangle$ iff $\langle A_1, A_0 \rangle \sqsubseteq \langle B_1, B_0 \rangle^{\sim}$.

(iii) $\Box \langle A_1, A_0 \rangle = \langle A_1, A_0 \rangle^{\perp \sim}$.

(iv) $\langle A_1, A_0 \rangle^* = \langle \varnothing, A_0 \rangle$.

(v) $\lambda = \langle \varnothing, W \rangle$.

One can prove that $\mathscr{F} := \langle W, \perp, \perp\!\!\!\!\sim, \Box, {}^*, \lambda \rangle$ is a BZ^3 frame. Take Ω as the set $P(\mathscr{F})$ of all $\perp$-regular subsets of $\mathscr{F}$. Define $\rho(\gamma) = \langle v(\gamma)]$, for any formula γ, where $\langle v(\gamma)]$ is the principal ideal determined by $v(\gamma)$. One can easily check that $\mathscr{K}^{\mathscr{M}}$ is a Kripke model s.t. $T \models_{\mathscr{M}^c} \alpha$ iff $T \models_{\mathscr{K}^{\mathscr{M}}} \alpha$. ■

Some problems concerning Brouwer–Zadeh logics remain open. In particular:

1. Is the BZ poset of all effects of an infinite-dimensional Hilbert space a de Morgan BZ poset?

2. Is every de Morgan BZ poset (lattice) embeddable into a complete de Morgan BZ lattice? If not, is the de Morgan BZ poset of all effects of a finite-dimensional Hilbert space embeddable into a complete de Morgan BZ poset?

3. Is there any Kripke characterization of the de Morgan BZ logic, i.e., a logic algebraically characterized by the class of all de Morgan BZ lattices? In this framework, the problem can be reformulated in this way: is the (strong) de Morgan law elementary?

4. Is it possible to axiomatize a logic based on an infinite many-valued generalization of the ortho-pair semantics?

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